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Ultradiscretization without positivity

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Abstract
We present a new ultradiscretization approach which can be applied to discrete systems, the solutions of which are not positive definite. This was made possible, thanks to an ansatz involving the hyperbolic-sine function. We apply this new procedure to simple mappings. For the linear and homographic mappings, we obtain ultradiscrete forms and explicitly construct their solutions. Two discrete Painlevé II equations are also analysed and ultradiscretized. We show how to construct the ultradiscrete analogues of their rational and Airy-type solutions.

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1. Introduction

Constructing generalized cellular automata from discrete systems is a procedure that has become systematic and straightforward, thanks to the ultradiscretization approach [1]. Starting from a given nonlinear discrete system one can, by applying the appropriate limit, obtain a piecewise linear system, which manages to preserve the essential properties of the initial one. Most notable among the latter are the characteristics of integrability (conserved quantities, coherent structures etc.) and thus one can confidently claim that the ultradiscretization procedure preserves integrability.

The way ultradiscretization works is by a change of variables and a subsequent limit. Starting from the variable x, we introduce a new one X through

$$x = e^{X/\epsilon} \tag{1.1}$$

where ϵ is a parameter, and then we take the limit of the expressions involved when $\epsilon \to +0$. As a result, the multiplications and additions that appear in the discrete equation involving the variable x are replaced by additions and max-functions operating on the variables X. The latter is the consequence of the well-known result

$$\lim_{\epsilon\to+0} \epsilon \log(e^{A/\epsilon} + e^{B/\epsilon}) = \max(A, B). \tag{1.2}$$

A few remarks are in order here:

(1) The ultradiscretization procedure as described here concerns rational expressions. Thus, if we have a non-rational equation, such as the sine-Gordon, we must first convert it to a rational form [2, 3].

(2) Since the result of the ultradiscretization is an equation involving only additions and max-functions, it is clear that if all parameters and initial conditions are integer, the solution will remain integer throughout [1].

(3) For the ultradiscretization to be applicable, the variables and the parameters of the initial system must take only positive values (otherwise (1.1) would not have a sense in the real domain).

The positivity requirement is a serious constraint, which limits the class of equations that can be studied by the ultradiscretization approach. Despite this restriction, a great number of interesting nonlinear systems are amenable to this treatment, as can be inferred from the abundant existing literature [3–8].

Still, the positivity constraint has been a serious obstacle to the extension of the ultradiscretization approach. In an attempt to overcome this difficulty, a different kind of ultradiscretization ansatz has been recently proposed, which can be equally applied to positive or negative variables [9]. Instead of (1.1), we introduce an ansatz involving the hyperbolic sine,

$$x = \sinh \frac{X}{\epsilon} \tag{1.3}$$

and then take the limit $\epsilon \to +0$. Clearly, for (1.3) to be applicable, the variable x does *not* have to be positive definite. In what follows, in order to distinguish between the two approaches, we shall refer to the one involving the exponential function as the e-ultradiscretization and the new one, based on the hyperbolic sine, as s-ultradiscretization.

For practical applications, the s-ultradiscretization can be obtained in a simple way by noting that $2\sinh(X/\epsilon) = \exp(X/\epsilon) - \exp(-X/\epsilon)$. As a matter of fact, the factor 2 which is introduced due to the definition of the hyperbolic sine can be omitted. This amounts to using

$$x = e^{X/\epsilon} - e^{-X/\epsilon} \tag{1.4}$$

instead of (1.3). Given this expression, it is straightforward to obtain the s-ultradiscrete limit of a given discrete equation.

Two points must be stressed here.

(1) Since the ansatz (1.4) is not monomial as in the case of (1.3), it is advised to convert all equations to polynomial form (by multiplying out the denominators) before applying the procedure. The consequence of this is that one often obtains equations which are implicit for the dependent variable.

Another difficulty might have stemmed from constants or independent variables which, though positive, are smaller than 1. In that case, introducing an ansatz involving a negative exponent would have been inconsistent. However, we can consider the inverse of this quantity, which is larger than 1, and introduce an ansatz with a positive exponent. For instance, if $a < 1$, we write $a = 1/e^{A/\epsilon}$. Next, we multiply out the denominator and then apply the limit $\epsilon \to 0$. However, this is not really necessary. In fact, proceeding as described and then subtracting A from both sides of the equation is equivalent to having written $a = e^{-A/\epsilon}$ in the first place, in which case the contribution would have been $-A$.

(2) A fundamental difficulty exists with the sinh transform. Let us assume that the variable x is expanded in terms of $\mathrm{e}^{-1/\epsilon}$ as

$$x = \mathrm{e}^{A/\epsilon} \sum_{k=0}^{\infty} c_k \, \mathrm{e}^{-k/\epsilon}. \tag{1.5}$$

Then, the inversion of (1.3) leads to

$$X = \lim_{\epsilon \to +0} \epsilon \operatorname{arcsinh} x = \operatorname{sgn}(c_0) \max(0, A) \tag{1.6}$$

i.e. only the dominant term gives a contribution. However, it may turn out that, when computing x, the actual value of A is smaller than the expected one, i.e. c_0 vanishes through cancellations. Then the result of the inversion of (1.3) will be $X = \operatorname{sgn}\big(c_{k_0}\big) \max(0, A - k_0)$, where k_0 corresponds to the first nonvanishing subdominant contribution. But there is no way a priori to know which is the first non-zero subdominant term, unless we have full knowledge of x (which is, in principle, given by the solution of a discrete equation). Thus, the s-ultradiscretization has by construction indeterminacies, which are due to the vanishing of the dominant contribution. In the applications, we shall consider in the following, we will discuss the appearance of such indeterminacies which obviously appear only at specific points of the evolution.

The equivalent of identity (1.2) in the case of an s-ultradiscretization can also be easily given (and we are going to assume here that the dominant contribution does exist). Starting from

$$\mathrm{e}^{X/\epsilon} - \mathrm{e}^{-X/\epsilon} = \sigma_\alpha \, \mathrm{e}^{A/\epsilon} + \sigma_\beta \, \mathrm{e}^{B/\epsilon} \tag{1.7}$$

(where $\sigma_\alpha^2 = \sigma_\beta^2 = 1$) we find that

$$X = \sigma_m \max(A, B, 0) \tag{1.8}$$

where σ_m is $\sigma_\alpha, \sigma_\beta$ or unity depending on which term is maximal. The extension of this identity to N terms is straightforward.

In this paper, we will examine simple one-dimensional discrete systems, derive their ultradiscrete form and obtain their solution. The systems we are going to focus on are linear mappings, discrete Riccati equations and discrete Painlevé equations. In all cases, we shall provide the e-ultradiscrete forms of the systems and their solutions as well as the s-ultradiscrete ones. The latter is of interest, of course, whenever the solution is not positive definite.

2. The linear mapping

In order to investigate the s- versus e-ultradiscretization approach, we will begin with the study of a very simple system: a first-order linear mapping. We start from

$$x_{n+1} = a x_n + b. \tag{2.1}$$

When a and b are functions of n, the solution of (2.1) can be given in a formal way, involving products and sums. While it is still possible to apply the ultradiscretization procedures to the equation and its solution, the result remains quite formal. Thus, in order to have an explicit example, we shall work with constant a and b. In this case, the solution to (2.1) can be simply written as

$$x_n = x_0 a^n + \frac{b}{a-1}(a^n - 1). \tag{2.2}$$

If we assume that a, b are positive, it is possible to perform the e-ultradiscretization of (2.1). We put $x = \mathrm{e}^{X/\epsilon}, a = \mathrm{e}^{A/\epsilon}, b = \mathrm{e}^{B/\epsilon}$ whereupon (2.1) becomes, at the limit $\epsilon \to +0$

$$X_{n+1} = \max(A + X_n, B). \tag{2.3}$$

The solution of (2.3) can be computed very easily,

$$X_n = \begin{cases} nA + \max(X_0, B - A), & \text{if} \quad A > 0 \\ nA + \max(X_0, B - nA), & \text{otherwise.} \end{cases} \tag{2.4}$$

The same result can be obtained from the e-ultradiscretization of (2.2). Substituting the expressions for x, a, b and taking the limit $\epsilon \to 0$, we obtain

$$X_n = \max(X_0 + nA, B + (n-1)\max(A, 0)) \tag{2.5}$$

which is identical to (2.4).

We now turn to the case of an s-ultradiscretization. Clearly, the variable x must take negative values for this case to be interesting, and we choose

$$x_{n+1} = -ax_n + b \tag{2.6}$$

where a, b are positive, and a minus sign has been explicitly introduced. (Other sign contributions would have been possible but their treatment is analogous to that of (2.6)). Since a, b are positive, the ansatz we shall use is $a = \mathrm{e}^{A/\epsilon}, b = \mathrm{e}^{B/\epsilon}$, even though A and B may be negative as explained in the introduction. However, for the variable x, we introduce $x = \mathrm{e}^{X/\epsilon} - \mathrm{e}^{-X/\epsilon}$. Substituting into (2.6), we have

$$\mathrm{e}^{X_{n+1}/\epsilon} - \mathrm{e}^{-X_{n+1}/\epsilon} = -\mathrm{e}^{A/\epsilon}(\mathrm{e}^{X_n/\epsilon} - \mathrm{e}^{-X_n/\epsilon}) + \mathrm{e}^{B/\epsilon}. \tag{2.7}$$

In order to apply the identity (1.2), we must have sums of exponential terms. Thus, we collect the terms with the same sign of the equality and take the limit $\epsilon \to 0$. We obtain thus:

$$\max(X_{n+1}, A + X_n) = \max(-X_{n+1}, A - X_n, B). \tag{2.8}$$

For the solutions of (2.8), we start from the discrete solution of (2.6) which is

$$x_n = \frac{b}{a+1} + (-1)^n a^n \left(x_0 - \frac{b}{a+1}\right). \tag{2.9}$$

We substitute the appropriate expressions for a, b, x and find

for $A \geqslant 0$:

$$\mathrm{e}^{X_n/\epsilon} - \mathrm{e}^{-X_n/\epsilon} \sim \mathrm{e}^{(B-A)/\epsilon} + (-1)^n\, \mathrm{e}^{nA/\epsilon}(\mathrm{e}^{X_0/\epsilon} - \mathrm{e}^{-X_0/\epsilon} - \mathrm{e}^{(B-A)/\epsilon}) \tag{2.10a}$$

for $A < 0$:

$$\mathrm{e}^{X_n/\epsilon} - \mathrm{e}^{-X_n/\epsilon} \sim \mathrm{e}^{B/\epsilon} + (-1)^n\, \mathrm{e}^{nA/\epsilon}(\mathrm{e}^{X_0/\epsilon} - \mathrm{e}^{-X_0/\epsilon} - \mathrm{e}^{B/\epsilon}). \tag{2.10b}$$

In order to extract X from (2.10), we shall use identity (1.7).

We find readily from (2.10), assuming $n > 0$,

for $A \geqslant 0$:

$$\mathrm{e}^{X_n/\epsilon} - \mathrm{e}^{-X_n/\epsilon} \sim (-1)^n\, \mathrm{e}^{nA/\epsilon}(\mathrm{sgn}(X_0)\, \mathrm{e}^{|X_0|/\epsilon} - \mathrm{e}^{(B-A)/\epsilon})$$

and finally

$$X_n = (-1)^n s(nA + \max(|X_0|, B - A)) \tag{2.11}$$

where $s = \mathrm{sgn}(X_0)$ if $|X_0| > B - A$ or $s = -1$ if $|X_0| < B - A$,
while for $A < 0$, $\mathrm{e}^{X_n/\epsilon} - \mathrm{e}^{-X_n/\epsilon} \sim \mathrm{e}^{B/\epsilon} + (-1)^n\, \mathrm{sgn}(X_0)\, \mathrm{e}^{(nA+|X_0|)/\epsilon}$ and finally

$$X_n = s\max(B, nA + |X_0|, 0) \tag{2.12}$$

where $s = 1$ if B or 0 are maximal and $s = (-1)^n \operatorname{sgn}(X_0)$ if $nA + |X_0|$ is maximal.

If $n < 0$, we can follow the same steps and obtain:

if $A \geqslant 0$:

$$X_n = s \max(B - A, nA + |X_0|, 0) \tag{2.13}$$

where $s = 1$ if $B - A$ or 0 are maximal and $s = (-1)^n \operatorname{sgn}(X_0)$ if $nA + |X_0|$ is maximal. Similarly, for $A < 0$, we find

$$X_n = (-1)^n s(nA + \max(|X_0|, B)) \tag{2.14}$$

where $s = \operatorname{sgn}(X_0)$ if $|X_0| > B$ or $s = -1$ if $|X_0| < B$.

This completes the construction of the solution of the s-ultradiscrete linear mapping. It is to be noted that other expressions should be introduced for specific initial values such as $X_0 = B - A$ for $A > 0$ and $n > 0$.

However, casting the s-ultradiscrete equation in the form of a max–equal–max mapping (2.8) (which is admittedly a little awkward to work with) is not something unavoidable in this case. Indeed, once we have equation (2.7), we can obtain an explicit equation for X_{n+1} using identity (1.8). We find

$$X_{n+1} = \sigma \max(A + X_n, A - X_n, B, 0) \tag{2.15}$$

where $\sigma = -1$ if all three conditions $X_n > 0$, $X_n + A > 0$ and $X_n + A - B > 0$ are satisfied, otherwise we have $\sigma = 1$. We note that (2.15) is valid unless X_n falls in 0 or $B - A$ for some n. At any rate, the X_{n+1} computed from (2.15) satisfies identically the max–equal–max equation (2.8).

3. The homographic mapping

The next system we shall examine is again first-order albeit a very special one, the homographic mapping. Among all its possible forms, we choose a simple one

$$x_{n+1} = a + \frac{b}{x_n} \tag{3.1}$$

where a, b are positive constants.

The e-ultradiscretization of this system is straightforward. Putting $x = \mathrm{e}^{X/\epsilon}, a = \mathrm{e}^{A/\epsilon}, b = \mathrm{e}^{B/\epsilon}$ we obtain, at $\epsilon \to 0$ the mapping

$$X_{n+1} = \max(A, B - X_n). \tag{3.2}$$

The discrete system (3.1) is linearizable through a Cole–Hopf transformation. Indeed, putting

$$x_n = \frac{q_{n+1}}{q_n} \tag{3.3}$$

we obtain for q the linear, second-order, mapping

$$q_{n+1} = aq_n + bq_{n-1}. \tag{3.4}$$

The ultradiscrete version of this transformation is simply ($q = \mathrm{e}^{Q/\epsilon}$)

$$X_n = Q_{n+1} - Q_n \tag{3.5}$$

and the equation for Q becomes

$$Q_{n+1} = \max(A + Q_n, B + Q_{n-1}). \tag{3.6}$$

Using Q defined by the latter, it is elementary to show that X, computed from (3.5), does indeed satisfy the ultradiscrete equation (3.2).

We turn now to the s-ultradiscretization of the homographic mapping and, as in section 2, we introduce an explicit minus sign

$$x_{n+1} = a - \frac{b}{x_n} \tag{3.7}$$

where a, b are again positive constants. A more convenient form of (3.7) is the following:

$$x_{n+1}x_n - (\lambda + \mu)x_n + \lambda\mu = 0 \tag{3.8}$$

where λ, μ are real and positive. The solution of (3.8) obtained through the Cole–Hopf transformation is

$$x_n = \frac{\lambda^{n+1} + c\mu^{n+1}}{\lambda^n + c\mu^n} \tag{3.9}$$

(corresponding to $q_n = \lambda^n + c\mu^n$) where c is an integration constant.

In order to s-ultradiscretize the mapping (3.8), we introduce $\lambda = \mathrm{e}^{\Lambda/\epsilon}$, $\mu = \mathrm{e}^{M/\epsilon}$ (where without loss of generality we can assume $\Lambda > M$) and $x = \mathrm{e}^{X/\epsilon} - \mathrm{e}^{-X/\epsilon}$. At the limit $\epsilon \to 0$, we obtain the equation

$$\max(|X_{n+1} + X_n|, \Lambda - X_n, M - X_n, \Lambda + M) = \max(|X_{n+1} - X_n|, \Lambda + X_n, M + X_n). \tag{3.10}$$

The Cole–Hopf transformation $q_{n+1} = x_n q_n$ becomes ($q = \mathrm{e}^{Q/\epsilon} - \mathrm{e}^{-Q/\epsilon}$)

$$\max(Q_{n+1}, |X_n - Q_n|) = \max(-Q_{n+1}, |X_n + Q_n|) \tag{3.11}$$

while the equation satisfied by Q is

$$\begin{aligned} &\max(Q_{n+1}, \Lambda - Q_n, M - Q_n, \Lambda + M + Q_{n-1}) \\ &\quad= \max(-Q_{n+1}, \Lambda + Q_n, M + Q_n, \Lambda + M - Q_{n-1}). \end{aligned} \tag{3.12}$$

Since the last two equations are implicit, it does not seem possible to verify formally that they give indeed only the solution of (3.10). Still, using the explicit solution (3.9) for x (and for q), one can show that these equations are indeed satisfied. We start by computing the s-ultradiscrete limit of (3.9). Putting $c = \mathrm{e}^{C/\epsilon} - \mathrm{e}^{-C/\epsilon}$, we have

$$\mathrm{e}^{X/\epsilon} - \mathrm{e}^{-X/\epsilon} = \mathrm{e}^{\Lambda/\epsilon} \frac{1 + (\mathrm{e}^{C/\epsilon} - \mathrm{e}^{-C/\epsilon})\,\mathrm{e}^{(M-\Lambda)(n+1)/\epsilon}}{1 + (\mathrm{e}^{C/\epsilon} - \mathrm{e}^{-C/\epsilon})\,\mathrm{e}^{(M-\Lambda)n/\epsilon}}. \tag{3.13}$$

Since we will be looking for the dominant part of the rhs, we can rewrite (3.13) keeping only the dominant contribution of C

$$\mathrm{e}^{X/\epsilon} - \mathrm{e}^{-X/\epsilon} \sim \mathrm{e}^{\Lambda/\epsilon} \frac{1 + \mathrm{sgn}(C)\,\mathrm{e}^{|C|/\epsilon + (M-\Lambda)(n+1)/\epsilon}}{1 + \mathrm{sgn}(C)\,\mathrm{e}^{|C|/\epsilon + (M-\Lambda)n/\epsilon}}. \tag{3.14}$$

It is now straightforward to follow the dominant contributions and write the solution.

If $|C| > (\Lambda - M)(n+1)$ the second terms in numerator and denominator are dominant. Keeping only these terms, we find for the rhs the result $\mathrm{e}^{M/\epsilon}$ and thus the solution is

$$X_n = \max(M, 0) \qquad \text{for} \quad |C| > (\Lambda - M)(n+1). \tag{3.15a}$$

In contrast, if $|C| < (\Lambda - M)n$ the first term is dominant resulting to $\mathrm{e}^{\Lambda/\epsilon}$ as only contribution. We have thus

$$X_n = \max(\Lambda, 0) \qquad \text{for} \quad |C| < (\Lambda - M)n. \tag{3.15b}$$

Finally, when $(\Lambda - M)(n+1) > |C| > (\Lambda - M)n$, the dominant contribution that comes from the combination of numerator and denominator is $\mathrm{sgn}(C)\,\mathrm{e}^{\Lambda/\epsilon - |C|/\epsilon - (M-\Lambda)n/\epsilon}$. We find that the solution is

$$X_n = \begin{cases} \mathrm{sgn}(C)(\Lambda - |C| + (\Lambda - M)n), & \text{if} \quad |C| - \Lambda < (\Lambda - M)n \\ 0, & \text{otherwise} \end{cases}$$
$$\text{for } (\Lambda - M)(n+1) > |C| > (\Lambda - M)n. \tag{3.15c}$$

It is noted that other expressions should again be introduced for specific initial values. For the quantity Q we have similarly

$$e^{Q/\epsilon} - e^{-Q/\epsilon} \sim e^{\Lambda n/\epsilon} + \operatorname{sgn}(C)\, e^{|C|/\epsilon + Mn/\epsilon}. \tag{3.16}$$

At the limit $\epsilon \to 0$ and applying the indentity (1.7) we find

$$Q_n = \sigma \max(\Lambda n, Mn + |C|, 0) \tag{3.17}$$

where

$$\sigma = \begin{cases} \operatorname{sgn}(C), & \text{if} \quad |C| > (\Lambda - M)n \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to show that Q given from (3.17) does indeed satisfy (3.12) and similarly X and Q do satisfy the Cole–Hopf relation (3.11). Thus, the s-ultradiscretization does preserve the basic properties of the homographic mapping.

4. Ultradiscretization of discrete Painlevé equations and their solutions

In this section, we shall investigate the s-ultradiscrete forms of q-discrete Painlevé equations and of their solutions. The q-discrete Painlevé equations are nonautonomous mappings in which the independent variable appears in exponential form $z_n = z_0\lambda^n$. For ultradiscretization, we introduce the ansatz $\lambda = e^{1/\epsilon}$, and together with the choice $z_0 = 1$ we have $z_n = e^{n/\epsilon}$. (A more precise ansatz would have been $\lambda = e^{\Lambda/\epsilon}$, $z_0 = e^{K/\epsilon}$, leading to $z_n = e^{(\Lambda n + K)/\epsilon}$, but this does not alter the argument presented below).

4.1. Rational solutions of q-P$_{\rm II}$

Here we shall examine the rational solutions of the q-P$_{\rm II}$

$$x_{n+1}x_{n-1} = \frac{z_n x_n + a}{\sigma x_n + z_n} \tag{4.1}$$

where $\sigma^2 = 1$. We start by taking $\sigma = +1$ in which case (4.1) can be e-ultradiscretized [10]. Putting $x = e^{X/\epsilon}$, $z_n = e^{n/\epsilon}$, $a = e^{A/\epsilon}$ we find

$$X_{n+1} + X_{n-1} = \max(X_n + n, A) - \max(X_n, n). \tag{4.2}$$

Equation (4.1) possesses rational solutions for special values of the parameter a. We have indeed $x = 1$ for $a = 1$, $x = (z + \lambda^3(1 + \lambda + \lambda^2))/(z + \lambda(1 + \lambda + \lambda^2))$ for $a = \lambda^6$ and higher solutions can be constructed for $a = \lambda^{6m}$ for integer m. The e-ultradiscrete form of these rational solutions can be easily constructed, either ultradiscretising the discrete solutions or working directly with (4.2). We find that the simplest solution is $X = 0$ for $A = 0$. The next solutions (easily verified) are $(\ldots, 2, 2, 1, 0, 0, \ldots)$ for $A = 6$, $(\ldots, 4, 4, 3, 2, 2, 2, 1, 0, 0, \ldots)$ for $A = 12$, $(\ldots, 6, 6, 5, 4, 4, 4, 3, 2, 2, 2, 1, 0, 0, \ldots)$ for $A = 18$, and so on. In general, for $A = 6m$ we have a solution which goes from a constant value $2m$ for $n \to -\infty$ to 0 for $n \to +\infty$, with $2m$ intermediate down-steps.

We turn now to the case $\sigma = -1$. This q-P$_{\rm II}$ has rational solutions, which present poles and zeros (except for the solution $x = 1$ obtained for $a = -1$). We have, for instance, $x = (z - \lambda^3(1 + \lambda + \lambda^2))/(z - \lambda(1 + \lambda + \lambda^2))$ for $a = -\lambda^6$. In order to s-ultradiscretize the $\sigma = -1$ q-P$_{\rm II}$, we multiply out all denominators, introduce the ansatz $x = e^{X/\epsilon} - e^{-X/\epsilon}$, $z_n = e^{n/\epsilon}$, $a = e^{A/\epsilon} - e^{-A/\epsilon}$ and find

$$\begin{aligned}(e^{X_{n+1}/\epsilon} - e^{-X_{n+1}/\epsilon})(e^{X_{n-1}/\epsilon} - e^{-X_{n-1}/\epsilon})(-e^{X_n/\epsilon} + e^{-X_n/\epsilon} + e^{n/\epsilon}) \\ = e^{n/\epsilon}(e^{X_n/\epsilon} - e^{-X_n/\epsilon}) + e^{A/\epsilon} - e^{-A/\epsilon}.\end{aligned} \tag{4.3}$$

This leads, at $\epsilon \to 0$, to the following ultradiscrete equation:

$$\max(|X_{n+1} + X_{n-1}| + \max(n, -X_n), |X_{n+1} - X_{n-1}| + X_n, n - X_n, -A)$$
$$= \max(|X_{n+1} + X_{n-1}| + X_n, |X_{n+1} - X_{n-1}| + \max(n, -X_n), n + X_n, A). \quad (4.4)$$

The first interesting solution of 'rational' type turns out to be $(\ldots, 2, 2, -1, 0, 0, \ldots)$ for $A = -6$. However, a closer inspection yields another solution of the same type $(\ldots, 2, -2, -1, 0, 0, \ldots)$ for the same value of A. The origin of these solutions is easy to understand, provided we use the more detailed ansatz $z_n = \mathrm{e}^{(n+K)/\epsilon}$ (with $1 > K > 0$) for the independent variable. In this case, the ultradiscretization of the simple rational solution leads to two negative values: $-1 - K, -K$. Thus, if one insists on a z_n of the form λ^n there exist two possibilities: either $K \to 0^+$ or $K \to 1^-$. In the first case, the negative values are $(-1, 0)$, while in the second we have $(-2, -1)$, which explains the two possibilities found above.

Similarly, for the higher solutions, we have two possibilities. For $A = 12$ we find the solutions $(\ldots, 4, -4, -3, 2, 2, -2, -1, 0, 0, \ldots)$ or $(\ldots, 4, 4, -3, -2, 2, 2, -1, 0, 0, \ldots)$. Similarly, for $A = 18$, there exist two possible solutions $(\ldots, 6, -6, -5, 4, 4, -4, -3, 2, 2, -2, -1, 0, 0, \ldots)$ and $(\ldots, 6, 6, -5, -4, 4, 4, -3, -2, 2, 2, -1, 0, 0, \ldots)$ and so on for higher A's.

Thus, at the s-ultradiscrete limit the zeros and poles of the discrete solution are manifesting themselves through the sign changes of the solution.

4.2. *Solutions of q-P_{II} in terms of the q-Airy function*

In order to investigate the solutions of q-$\mathrm{P_{II}}$ which involve the discrete logarithmic derivative of the q-Airy function we consider the q-$\mathrm{P_{II}}$

$$x_{n+1}x_{n-1} = \frac{z_n x_n - 1}{x_n(a z_n - x_n)}. \quad (4.5)$$

We seek a solution of (4.5) through linearization, using the methods developed in [11]. We find that when $a = 1/\lambda$ there exists a solution given by the homographic mapping

$$x_{n+1} = \frac{z_n x_n - 1}{x_n} \quad (4.6)$$

which can be linearized through the Cole–Hopf transformation $x_n = q_n/q_{n-1}$. We obtain thus for q one of the forms of the q-Airy equation,

$$q_{n+1} - z_n q_n + q_{n-1} = 0. \quad (4.7)$$

We proceed now to the s-ultradiscretization of the q-$\mathrm{P_{II}}$ and its solution. Putting $x = \mathrm{e}^{X/\epsilon} - \mathrm{e}^{-X/\epsilon}$, $z_n = \mathrm{e}^{n/\epsilon}$, $a = \mathrm{e}^{A/\epsilon} - \mathrm{e}^{-A/\epsilon}$ we find, at the limit $\epsilon \to 0$, the ultradiscrete equation,

$$\max(|X_{n+1} + X_n + X_{n-1} + A|, |X_{n+1} - X_n + X_{n-1} - A|, |-X_{n+1} + X_n + X_{n-1} - A|,$$
$$|X_{n+1} + X_n - X_{n-1} - A|, |X_{n+1} - 2X_n - X_{n-1}| - n, |X_{n+1} + 2X_n - X_{n-1}| - n,$$
$$|X_{n+1} + X_{n-1}| - n, -X_n) = \max(|X_{n+1} + X_n + X_{n-1} - A|, |X_{n+1} - X_n + X_{n-1}$$
$$+ A|, |-X_{n+1} + X_n + X_{n-1} + A|,$$
$$|X_{n+1} + X_n - X_{n-1} + A|, |X_{n+1} - 2X_n + X_{n-1}| - n, |X_{n+1} + 2X_n + X_{n-1}| - n,$$
$$|X_{n+1} - X_{n-1}| - n, X_n). \quad (4.8)$$

Similarly, the q-Airy equation can be ultradiscretized, by $q = \mathrm{e}^{Q/\epsilon} - \mathrm{e}^{-Q/\epsilon}$ leading to

$$\max(Q_{n+1}, n - Q_n, Q_{n-1}) = \max(-Q_{n+1}, n + Q_n, -Q_{n-1}). \quad (4.9)$$

We start by constructing the solution of s-ultradiscrete Airy equation (4.9). For large positive n, the dominant term on the rhs is $Q_n + n$ and the equation reduces to $Q_{n+1} = Q_n + n$, with solution $Q_n = n(n-1)/2 + c$. For large negative n on the other hand, we obtain the equation $Q_{n+1} + Q_{n-1} = 0$. Its solution has a period-4 involving two arbitrary constants $Q_n = \alpha \mathrm{i}^n + \alpha^*(-\mathrm{i})^n$ and is better given in the form $(\ldots, a, b, -a, -b, a, b, \ldots)$. It remains now to match the two solutions. This depends on the values of a and b. For negative n, we find $(\ldots, 0, 1, 0, -1, 0, 1, \ldots)$. The sequence breaks down at $Q_1 = 1$ (instead of $Q_1 = -a = 0$) and matching the solutions for positive n we obtain $c = 1$.

The Cole–Hopf relation leads to the s-ultradiscrete equation,

$$\max(Q_n, |X_n - Q_{n-1}|) = \max(-Q_n, |X_n + Q_{n-1}|). \tag{4.10}$$

We can now construct the solution for X, since Q is known. For negative n, assuming $a > b > 0$, we obtain the sequence $(\ldots, 0, b-a, 0, b-a, 0, b-a, \ldots)$, while if $b > a > 0$ we find $(\ldots, b-a, 0, b-a, 0, b-a, 0, \ldots)$, i.e. the same formal solution. We must point out here that while the solution for Q involved two arbitrary constants, only one combination survives in X, as expected. For positive n given that Q is large and positive, the equation reduces to $X_n = Q_n - Q_{n-1}$ leading to $X_n = n - 1$. Thus, the s-ultradiscrete form of the Airy-type solution of q-$\mathrm{P_{II}}$ is a saw-tooth function on the left matched to a linear one on the right.

This solution can be directly obtained from the s-ultradiscrete form of the homographic mapping (4.6)

$$\max(|X_{n+1} + X_n|, n - X_n) = \max(|X_{n+1} - X_n|, n + X_n). \tag{4.11}$$

For large negative n, the equation is just $|X_{n+1} + X_n| = |X_{n+1} - X_n|$ leading to a period-2 X alternating between a finite value and zero, while for positive n the equation reduces $|X_{n+1} + X_n| = X_n + n$ and finally $X_{n+1} = n$. In the specific example of $a = 0, b = 1$ we have given above, the solution is $(\ldots, 1, 0, 1, 0, 1, \ldots)$ for negative n, $X_0 = 1$, matched to $X_n = n - 1$ for positive n.

Thus, with this procedure, we can construct an s-ultradiscrete analogue to a q-Painlevé equation and obtain its solution in terms of the s-ultradiscrete analogue of a special function.

5. Conclusion

Why is ultradiscretization important? The main answer is that it is the only *systematic* method which can produce a cellular-automaton analogue of a given equation while preserving its fundamental properties and in particular integrability. The procedure entails several constraints. First, we can only apply it to a discrete equation. Thus, if we wish to find the cellular-automaton analogue of a given evolution equation, we must first construct its discrete form. Of course, this is not a serious constraint: discrete systems are the most fundamental entities and a decade of intense activity has produced the discrete analogues of most well-known soliton equations. The second requirement of the ultradiscretization procedure, as first introduced, is that all quantities be positive definite. The aim of the present work is precisely to introduce a new procedure, which does not rely on positivity. The third constraint has to do with nonautonomous systems. Since the ultradiscretization consists in taking a limit where, schematically, the variables are replaced by their logarithms, for the independent variable to survive ultradiscretization it must be exponential. We are limited thus to q-type equations. However, this is not a serious constraint either since, as our studies on discrete Painlevé equations have shown, q forms exist along all the difference forms and are in some sense more fundamental.

In this paper, we have introduced an ultradiscretization ansatz involving the hyperbolic sine which does away with positivity requirements. This extra freedom comes of course at some price. As we explained in the introduction, when the dominant contribution happens to vanish, the result is indeterminate and its determination would have required a more precise knowledge of the solution of the underlying discrete system. Moreover, the resulting equations are usually in the form of max–equal–max equations which, being implicit, are not very convenient for the evolution.

In this study, we have limited ourselves to simple one-dimensional mappings. For the linear and homographic mappings, we have obtained their ultradiscrete forms and their explicit solutions. In the case of discrete Painlevé equations, we have studied two particular forms of q-$\mathrm{P_{II}}$. For the first, we have given the ultradiscrete form of the rational solution which presents zeros and poles: the traces of the latter are present through sign changes in the ultradiscrete solution. Still more interesting is the result obtained for the second q-$\mathrm{P_{II}}$. We have obtained the s-ultradiscrete form of the particular solution involving the q-Airy function (something impossible with the previous ultradiscretization procedure). It would be interesting to extend the s-ultradiscretization approach to the other q-Painlevé equations in order to study their special solutions in terms of special functions of the q-hypergeometric family.

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